Lecture 3b on Standard Model Effective Field Theory

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Outline

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RG running at one loop

- Conventional perturbation theory may fail for a process that involves large ratios of scales, e.g., m/M, since $(g/(4\pi))^2 \ln(M/m)$ could be large.
  m: typical external particle mass/momentum,
  M: internal particle mass.
- This issue can be best handled in EFT:
  log-independent term by matching and
  log enhancement by RG running.
  Matching and RG running can be done independently and at different orders as required.

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RG running at one loop

- Parameters do not exhibit scale dependence at tree level, though matching is done at M. – This is a loop effect.
- In matching calculation, same renormalization scheme must be applied in UV and IR theories.
  The integrated-out heavy field offers the only scale M. Thus large log can be avoided in matching by setting renormalization/matching scale $\mu = M$.
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RG running at one loop – Example 1

Example 1: One-loop RG running of G_S in $\mathscr{L}_2(\phi,\psi)$

- We use mass-independent renormalization scheme: dimensional regularization (DR) with minimal subtraction (MS)
- To do renormalization, consider $\mathscr{L}_2(\phi,\psi)$ in terms of bare quantities:

$$\mathscr{L}_2(\phi,\psi) = i\bar{\psi}_0\partial\!\!\!/\psi_0 + \frac{1}{2}G_S^0\bar{\psi}_0\psi_0\bar{\psi}_0\psi_0 - y_\phi^0\bar{\psi}_0\psi_0\phi_0 + \text{terms not relevant here} \quad (1)$$

- In $d = 4 - 2\varepsilon$ dimensions, the dimensions of fields are modified to

$$[\psi] = \frac{3}{2} - \varepsilon,\ [\phi] = 1 - \varepsilon \quad (2)$$

Fields and couplings are renormalized as

$$\psi_0 = \sqrt{Z_\psi}\psi,\ \phi_0 = \sqrt{Z_\phi}\phi,\ G_S^0 = Z_{G_S}\mu^{2\varepsilon}G_S,\ y_\phi^0 = Z_{y_\phi}\mu^{\varepsilon}y_\phi \quad (3)$$

where an arbitrary mass scale μ is introduced so that all renormalized parameters reserve their dimensions in 4 dim.

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- Zs deviate from unity because of quantum effects.
  In perturbation theory $Z-1$ is considered small. Here at one loop $Z-1 \propto y_\phi^2$.
- Thus $\mathscr{L}_2(\phi,\psi)$ splits into a renormalized piece and counterterm (c.t.) piece:

$$\mathscr{L}_2(\phi,\psi) = iZ_\psi \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}Z_{G_S}Z_\psi^2 G_S \mu^{2\varepsilon}\bar{\psi}\psi\bar{\psi}\psi - Z_{y_\phi}Z_\psi Z_\phi^{1/2} y_\phi \mu^\varepsilon \bar{\psi}\psi\phi + \cdots \tag{4}$$

$$\begin{aligned} &= i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}G_S\mu^{2\varepsilon}\bar{\psi}\psi\bar{\psi}\psi - y_\phi\mu^\varepsilon\bar{\psi}\psi\phi \\ &\quad + i\left[Z_\psi - 1\right]\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\left[Z_{G_S}Z_\psi^2 - 1\right]G_S\mu^{2\varepsilon}\bar{\psi}\psi\bar{\psi}\psi \\ &\quad - \left[Z_{y_\phi}Z_\psi Z_\phi^{1/2} - 1\right]y_\phi\mu^\varepsilon\bar{\psi}\psi\phi + \cdots \end{aligned} \tag{5}$$

c.t.: determined by renormalization conditions, and thus scheme dependent.
In MS, they contain only UV divergent terms.

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&= i\bar\psi \partial\!\!\!/ \psi + \frac{1}{2} G_S \mu^{2\varepsilon} \bar\psi\psi\bar\psi\psi - y_\phi \mu^\varepsilon \bar\psi\psi\phi \\
&\quad + i\left[Z_\psi - 1\right] \bar\psi \partial\!\!\!/ \psi + \frac{1}{2}\left[Z_{G_S} Z_\psi^2 - 1\right] G_S \mu^{2\varepsilon} \bar\psi\psi\bar\psi\psi \\
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RG running at one loop – Example 1

- To compute RG equations (RGE) for G_S , start from the fact that bare quantities are independent of μ:

$$0 = \mu \frac{dG_S^0}{d\mu} = Z_{G_S} \mu^{2\varepsilon} \mu \frac{dG_S}{d\mu} + \mu^{2\varepsilon} G_S \mu \frac{dZ_{G_S}}{d\mu} + 2\varepsilon Z_{G_S} \mu^{2\varepsilon} G_S \quad (6)$$

$$\Rightarrow \quad \beta_{G_S} = \mu \frac{dG_S}{d\mu} = -G_S \mu \frac{d \ln Z_{G_S}}{d\mu} - 2\varepsilon G_S \quad (7)$$

  In mass-independent schemes, μ dependence enters only through couplings.

- Here we are computing RGE for G_S due to Yukawa coupling y_ϕ of ϕ with ψ, i.e.,

$$\mu \frac{d \ln Z_{G_S}}{d\mu} \propto \mu \frac{dy_\phi}{d\mu} = \beta_{y_\phi} \quad (8)$$

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- β_{y_ϕ} can be manipulated as for β_{G_S}:

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$$\Rightarrow \quad \beta_{y_\phi} = -y_\phi \mu \frac{d \ln Z_{y_\phi}}{d\mu} - \varepsilon y_\phi \quad (10)$$

Again, the first term is of higher order than the second term, and can be dropped for our purpose here.

- In summary, the leading term is

$$\beta_{G_S} = -G_S \mu \frac{dy_\phi}{d\mu} \frac{d \ln Z_{G_S}}{dy_\phi} - 2\varepsilon G_S = -G_S \beta_{y_\phi} \frac{d \ln Z_{G_S}}{dy_\phi} - 2\varepsilon G_S \quad (11)$$

$$\Rightarrow \quad \beta_{G_S} = \lim_{\varepsilon \to 0} \left[-G_S(-\varepsilon y_\phi) \frac{d \ln Z_{G_S}}{dy_\phi} \right] \quad (12)$$

Thus, to get RG running of G_S, we have to determine Z_{G_S}.

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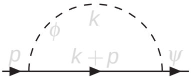
RG running at one loop – Example 1

- Cautions:
  Renormalized quantities are regular in the limit $\varepsilon \to 0$.
  The limit can only be properly taken in the end of calculation.
- But to get z_{G_S}, we also need z_ψ. Easiest thing first: z_ψ.



We need the term $\propto \not{p}$:

$$\begin{aligned} \text{diagram} &= \int \frac{d^dk}{(2\pi)^d} (-iy_\phi \mu^\varepsilon) \frac{i}{\not{k}+\not{p}} (-iy_\phi \mu^\varepsilon) \frac{i}{k^2-m^2} \\ &= y_\phi^2 \mu^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}+\not{p}}{(k+p)^2(k^2-m^2)} \end{aligned}$$

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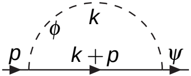


We need the term $\propto \not{p}$:

$$\text{diagram} = \int \frac{d^d k}{(2\pi)^d} (-iy_\phi \mu^\varepsilon) \frac{i}{\not{k}+\not{p}} (-iy_\phi \mu^\varepsilon) \frac{i}{k^2 - m^2}$$

$$= y_\phi^2 \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\not{k}+\not{p}}{(k+p)^2 (k^2 - m^2)}$$

RG running at one loop – Example 1

Use Feynman parameter x to combine the two denominators:

$$\frac{1}{(k+p)^2(k^2-m^2)} = \int_0^1 dx\, \frac{1}{[x(k+p)^2+(1-x)(k^2-m^2)]^2}$$

$$= \int_0^1 dx\, \frac{1}{[\ell^2-\Delta+i0^+]^2}, \; \ell = k+xp, \; \Delta = m^2(1-x)-p^2x(1-x)$$

Replace $k = \ell - xp$:

$$\text{diagram} = y_\phi^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\not\ell + (1-x)\not p}{[\ell^2-\Delta+i0^+]^2}$$

$$= \not p y_\phi^2 \int_0^1 dx\, (1-x)\, \mu^{2\varepsilon} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2-\Delta+i0^+]^2}$$

RG running at one loop – Example 1

Use standard loop integrals in $d = 4 - 2\varepsilon$ dims:

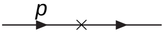
$$\mu^{2\varepsilon} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta + i0^+]^2} = \frac{i}{(4\pi)^2} \left[\frac{4\pi\mu^2}{\Delta}\right]^\varepsilon \Gamma(\varepsilon)$$

$$= \frac{i}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{\Delta} + O(\varepsilon)\right] \tag{13}$$

We finally get

$$\text{diagram} = \not{p} y_\phi^2 \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \int_0^1 dx\, (1-x) + \text{finite} = \not{p} y_\phi^2 \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \frac{1}{2} + \text{finite}$$

Requiring the c.t. diagram



$$= i(Z_\psi - 1)\not{p}$$

to cancel the UV divergent term (MS), we obtain

$$i(Z_\psi - 1)\not{p} + \not{p} y_\phi^2 \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \frac{1}{2} = 0 \Rightarrow (Z_\psi - 1) = -\frac{1}{2} \frac{y_\phi^2}{(4\pi)^2} \frac{1}{\varepsilon} \tag{14}$$

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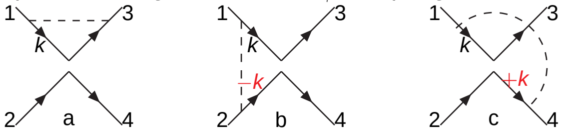
- Now we compute z_{G_S} .
  The one-loop $\bar{\psi}\psi\bar{\psi}\psi$ diagrams due to y_ϕ couplings in EFT_2 are



- Focus on $\bar{\psi}(p_3)\psi(p_1)\bar{\psi}(p_4)\psi(p_2)$, ignoring trivial crossing for both one-loop diagrams and c.t.
- These diagrams are at most logarithmically divergent.
- We are interested only in divergent terms which are independent of external momenta.
  We can thus set $p_i = 0$.

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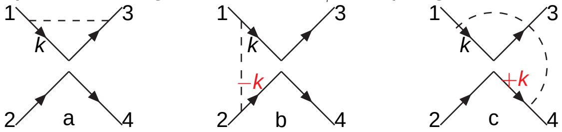

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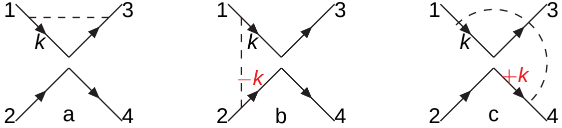

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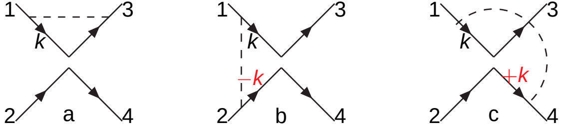

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The diagrams give

$$\begin{aligned}
\text{diag a} &= 2\times iG_S\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\bar{u}_3(-iy_\phi\mu^\varepsilon)\frac{i}{\not{k}}\frac{i}{\not{k}}(-iy_\phi\mu^\varepsilon)u_1\frac{i}{k^2-m^2}\bar{u}_4u_2 \\
&= 2iG_S\mu^{2\varepsilon}\left(\bar{u}_3u_1\bar{u}_4u_2\right)\times iy_\phi^2\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k^2-m^2)} \\
&= 2iG_S\mu^{2\varepsilon}\left(\bar{u}_3u_1\bar{u}_4u_2\right)\times\frac{-y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon}+\text{finite} \qquad (15)\\
\text{diag b} &= 2\times iG_S\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\bar{u}_3\frac{i}{\not{k}}(-iy_\phi\mu^\varepsilon)u_1\bar{u}_4\frac{i}{-\not{k}}(-iy_\phi\mu^\varepsilon)u_2\frac{i}{k^2-m^2} \qquad (16)\\
\text{diag c} &= 2\times iG_S\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\bar{u}_3\frac{i}{\not{k}}(-iy_\phi\mu^\varepsilon)u_1\bar{u}_4\frac{i}{+\not{k}}(-iy_\phi\mu^\varepsilon)u_2\frac{i}{k^2-m^2} \qquad (17)
\end{aligned}$$

diagrams b and c cancel each other!

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- *If* diagrams b and c did not cancel, they would induce a new structure

  $$\bar{u}_3 \gamma_\mu u_1 \; \bar{u}_4 \gamma^\mu u_2$$

  corresponding to the effective int. involving the dim-6 operator:

  $$\mathscr{O}_V = \bar{\psi}\gamma_\mu \psi \bar{\psi}\gamma^\mu \psi \tag{18}$$

  This is called mixing of operators under renormalization.

- Back to the issue. UV div in diag abc is required to cancel c.t.:

  $$\text{c.t. diagram for } \bar{\psi}\psi\bar{\psi}\psi = i(Z_{G_S} Z_\psi^2 - 1) G_S \mu^{2\varepsilon} (\bar{u}_3 u_1 \bar{u}_4 u_2) + \text{crossing} \tag{19}$$

  $$\Rightarrow \quad i(Z_{G_S} Z_\psi^2 - 1) G_S \mu^{2\varepsilon} + 2i G_S \mu^{2\varepsilon} \frac{-y_\phi^2}{(4\pi)^2} \frac{1}{\varepsilon} = 0$$

  $$\Rightarrow \quad (Z_{G_S} Z_\psi^2 - 1) = 2 \frac{y_\phi^2}{(4\pi)^2} \frac{1}{\varepsilon} \tag{20}$$

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- In perturbation theory $z-1$ is considered small though it may contain $1/\varepsilon$, because $\varepsilon \to 0$ is taken only in the end of calculation.

$$\begin{aligned} Z_{G_S} &= \left[1+2\frac{y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon}\right]\left[1+(Z_\psi-1)\right]^{-2} \\ &\approx 1+2\frac{y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon}-2(Z_\psi-1)=1+\frac{3y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon} \end{aligned} \tag{21}$$

- After this lengthy calculation, we obtain at one-loop level:

$$\beta_{G_S}=G_S\lim_{\varepsilon\to 0}\left[\varepsilon y_\phi\frac{d\ln Z_{G_S}}{dy_\phi}\right]\approx G_S\lim_{\varepsilon\to 0}\left[\varepsilon y_\phi\frac{dZ_{G_S}}{dy_\phi}\right]=G_S\frac{6y_\phi^2}{(4\pi)^2} \tag{22}$$

and RGE for G_S exact to one loop becomes

$$\mu\frac{dG_S}{d\mu}=\frac{6y_\phi^2}{(4\pi)^2}G_S \tag{23}$$

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$$\mu\frac{dG_S}{d\mu}=\frac{6y_\phi^2}{(4\pi)^2}G_S \tag{23}$$

RG running at one loop – Example 1

- Are we done?
  Not really. There is also μ dependence in coupling y_ϕ.
  If we ignore it together with μ dependence in G_S on rhs, we get in the so-called leading log approximation:

$$G_S(\mu) - G_S(M) = \frac{6y_\phi^2}{(4\pi)^2} G_S(M) \ln\frac{\mu}{M} \tag{24}$$

- We can do better by including μ dependence on rhs of RGE.
  For this we need the β function for y_ϕ, again due to y_ϕ interaction.
- Exercise – verify that

$$\beta_{y_\phi} = 5\frac{y_\phi^3}{(4\pi)^2} \tag{25}$$

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- Hints – Find first

$$Z_\phi - 1 = -2\frac{y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon},\ Z_{y_\phi} Z_\psi Z_\phi^{1/2} - 1 = \frac{y_\phi^2}{(4\pi)^2}\frac{1}{\varepsilon} \Rightarrow \quad Z_{y_\phi} - 1 = \tfrac{5}{2}\tfrac{y_\phi^2}{(4\pi)^2}\tfrac{1}{\varepsilon}$$

Then, proceed as follows

$$\beta_{y_\phi} = -y_\phi \mu \frac{d \ln Z_{y_\phi}}{d\mu} - \varepsilon y_\phi = -y_\phi \beta_{y_\phi} \frac{d \ln Z_{y_\phi}}{d y_\phi} - \varepsilon y_\phi$$

$$\Rightarrow \quad \beta_{y_\phi} = \lim_{\varepsilon \to 0} (-y_\phi)(-\varepsilon y_\phi)\frac{d \ln Z_{y_\phi}}{d y_\phi} = 5\frac{y_\phi^3}{(4\pi)^2} \tag{26}$$

- Important –

  Everything is manipulated for $\varepsilon \neq 0$ and in the spirit of pert. theory
  Only at the end of the day we take $\varepsilon \to 0$ for renormalizaed quantities.

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- Comments:
  1. β functions depend on renormalization schemes applied, mass dependent or independent.
  - In mass-dependent schemes βs vary smoothly in scale. See A. Manohar, arXiv:hep-ph/9606222.
  - In mass-independent schemes βs jump when crossing threshold of a heavy particle which is to be integrated out.
  - Although physical results are independent of schemes, mass-independent ones better suit the need of modern QFT: simpler topology of diagrams though more divergent; only UV divergence required for βs vs finite terms required in mass-dependent schemes.

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2. There is no essential difference in computing RGE between renormalizable and nonrenormalizable couplings. –
EFTs behave at low energies as well as renormalizable ones!

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- The above is very in QFT. We solve more generally the following:

$$\begin{cases} \mu \dfrac{d\ln G}{d\mu} = ag^2 \\ \mu \dfrac{dg^2}{d\mu} = b(g^2)^2 \end{cases} \qquad \left(G \to G_S,\, g \to y_\phi;\, a \to \frac{6}{(4\pi)^2},\, b \to \frac{10}{(4\pi)^2}\right) \tag{27}$$

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$$\frac{d\ln G}{dg^2} = \frac{a}{b}\frac{1}{g^2} \Rightarrow d\ln G = \frac{a}{b} d\ln(g^2)$$

$$\Rightarrow \quad \ln\frac{G(\mu_1)}{G(\mu_2)} = \frac{a}{b}\ln\frac{g^2(\mu_1)}{g^2(\mu_2)} \Rightarrow \frac{G(\mu_1)}{G(\mu_2)} = \left[\frac{g^2(\mu_1)}{g^2(\mu_2)}\right]^{a/b} \tag{28}$$

Summation of leading log to all orders!

- Exercise – verify that expansion of the above to leading order in $g^2(\mu_2)\ln(\mu_1/\mu_2)$ recovers the previous result in leading-log approximation. Hint – first solve $g^2(\mu)$ from its RGE.

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RG running at one loop – Example 2

Example 2: Mixing of operators under renormalization

Operators of same dim and symmetry can mix under renormalization.

- To see this in a simple framework, consider the EFT of ϕ, ψ:

$$\mathscr{L}_{\text{EFT}}(\phi,\psi) = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}G_V\mathscr{O}_V + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - y_\phi\bar{\psi}\psi\phi + \cdots, \quad (29)$$

$$\mathscr{O}_V = \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi, \quad (30)$$

where the effective interaction $G_V\mathscr{O}_V/2$ may have arisen from integrating out a heavy vector boson of mass M similarly to the case of 4-Fermi weak interactions.

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RG running at one loop – Example 2

- It turns out that its running is not closed!
  It induces at one loop a new interaction proportional to

$$\mathcal{O}_T = \bar{\psi}\sigma^{\mu\nu}\psi\bar{\psi}\sigma_{\mu\nu}\psi \tag{31}$$

- In principle other forms can also join mixing at higher orders.
  We work at one loop where O_V and O_T are closed under renor.
- Consistency therefore requires that we include both operators:

$$\mathcal{L}_{\mathrm{EFT}}(\phi,\psi) = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}G_V\mathcal{O}_V + \frac{1}{2}G_T\mathcal{O}_T + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - y_\phi\bar{\psi}\psi\phi + \cdots, \tag{32}$$

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RG running at one loop – Example 2

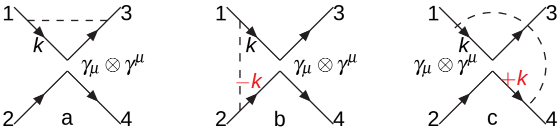
- Introduce c.t. as before to both interactions:

$$\mathscr{L}_{\mathrm{EFT}}(\phi,\psi) \supset +\frac{1}{2}\left[Z_{G_V} Z_\psi^2 - 1\right] G_V \mu^{2\varepsilon} \mathscr{O}_V + \frac{1}{2}\left[Z_{G_T} Z_\psi^2 - 1\right] G_T \mu^{2\varepsilon} \mathscr{O}_T \tag{33}$$

Z_ψ was known previously.

c.t.	to cancel UV div with one insertion of	that can induce an
$\left[Z_{G_V} Z_\psi^2 - 1\right] G_V \mu^{2\varepsilon}$	either $\mathscr{O}_V$ or $\mathscr{O}_T$	$\mathscr{O}_V$
$\left[Z_{G_T} Z_\psi^2 - 1\right] G_T \mu^{2\varepsilon}$	either $\mathscr{O}_V$ or $\mathscr{O}_T$	$\mathscr{O}_T$

Insertion of $G_V \mathscr{O}_V$:



RG running at one loop – Example 2

We set $p_i = 0$, and it is not necessary to include crossing diagrams.

$$\begin{aligned}\text{diagram a} &= 2\times iG_V\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\bar{u}_3(-iy_\phi\mu^\varepsilon)\frac{i}{\not k}\gamma_\mu\frac{i}{\not k}(-iy_\phi\mu^\varepsilon)u_1\frac{i}{k^2-m^2}\bar{u}_4\gamma^\mu u_2\\ &= -2G_V\mu^{2\varepsilon}y_\phi^2\left(\bar{u}_3\gamma_\alpha\gamma_\mu\gamma_\beta u_1\right)\left(\bar{u}_4\gamma^\mu u_2\right)\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta}{(k^2)^2(k^2-m^2)}\end{aligned} \quad (34)$$

Using symmetric loop integration,

$$\begin{aligned}&\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta}{(k^2)^2(k^2-m^2)} = \frac{1}{d}g^{\alpha\beta}\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)(k^2-m^2)}\\ &= \frac{1}{4}g^{\alpha\beta}\frac{i}{(4\pi)^2}\frac{1}{\varepsilon}+\text{finite}\end{aligned} \quad (35)$$

and

$$\gamma_\alpha\gamma_\mu\gamma_\beta g^{\alpha\beta} = (2-d)\gamma_\mu = (-2+2\varepsilon)\gamma_\mu \quad (36)$$

RG running at one loop – Example 2

we have finally

$$\text{diagram a} = -2G_V\mu^{2\varepsilon}y_\phi^2(-2)(\gamma_\mu\otimes\gamma^\mu)\frac{1}{4}\frac{i}{(4\pi)^2}\frac{1}{\varepsilon}+\text{finite}, \qquad (37)$$

where we denote $(A\otimes B)\equiv\bar{u}_3Au_1\bar{u}_4Bu_2$. The other two diagrams are

$$\begin{aligned}\text{diagram b} &= 2\times iG_V\mu^{2\varepsilon}(-iy_\phi\mu^\varepsilon)^2\int\frac{d^dk}{(2\pi)^d}\gamma_\mu\frac{i}{\not{k}}\otimes\gamma^\mu\frac{i}{-\not{k}}\frac{i}{k^2-m^2}\\ &= 2G_V\mu^{2\varepsilon}y_\phi^2(\gamma_\mu\gamma_\alpha\otimes\gamma^\mu\gamma^\beta)\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k_\beta}{(k^2)^2(k^2-m^2)}\\ &= +2iG_V\mu^{2\varepsilon}y_\phi^2(\gamma_\mu\gamma_\alpha\otimes\gamma^\mu\gamma^\alpha)\frac{1}{4}\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}+\text{finite} \qquad (38)\\ \text{diagram c} &= -2iG_V\mu^{2\varepsilon}y_\phi^2(\gamma_\alpha\gamma_\mu\otimes\gamma^\mu\gamma^\alpha)\frac{1}{4}\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}+\text{finite} \qquad (39)\end{aligned}$$

RG running at one loop – Example 2

It is nice that they sum to a tensor form:

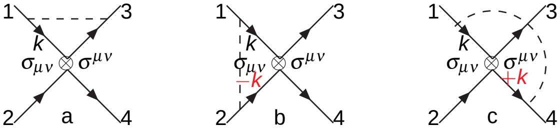
$$(\gamma_\mu\gamma_\alpha\otimes\gamma^\mu\gamma^\alpha)-(\gamma_\alpha\gamma_\mu\otimes\gamma^\mu\gamma^\alpha)=-i2(\sigma_{\mu\nu}\otimes\gamma^\mu\gamma^\nu)=-2(\sigma_{\mu\nu}\otimes\sigma^{\mu\nu}) \tag{40}$$

In summary,

$$\begin{aligned}&\text{diagrams with } C_V\mathscr{O}_V \text{ inserted}\\ =\quad & iG_V\mu^{2\epsilon}y_\phi^2\Big((\gamma_\mu\otimes\gamma^\mu)-(\sigma_{\mu\nu}\otimes\sigma^{\mu\nu})\Big)\frac{1}{(4\pi)^2}\frac{1}{\epsilon}+\text{finite}\end{aligned} \tag{41}$$

Note that mixing of operators takes place.

With an insertion of $G_T\mathscr{O}_T$, the diagrams are similar:



RG running at one loop – Example 2

But the algebra is more complicated. The diagrams yield

$$a = 2iG_T\mu^{2\varepsilon}y_\phi^2(-i)^2i^3\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{\not k}\sigma^{\mu\nu}\frac{1}{\not k}\otimes\sigma_{\mu\nu} = 0, \quad (42)$$

$$\begin{aligned} b &= 2iG_T\mu^{2\varepsilon}y_\phi^2(-i)^2i^3\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\sigma^{\mu\nu}\frac{1}{\not k}\otimes\sigma_{\mu\nu}\frac{1}{-\not k} \\ &= +2G_T\mu^{2\varepsilon}y_\phi^2\sigma^{\mu\nu}\gamma^\alpha\otimes\sigma_{\mu\nu}\gamma_\alpha\frac{i}{(4\pi)^2}\frac{1}{4}\frac{1}{\varepsilon}+\text{finite}, \end{aligned} \quad (43)$$

$$\begin{aligned} c &= 2iG_T\mu^{2\varepsilon}y_\phi^2(-i)^2i^3\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{\not k}\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\frac{1}{\not k} \\ &= -2G_T\mu^{2\varepsilon}y_\phi^2\gamma^\alpha\sigma^{\mu\nu}\otimes\sigma_{\mu\nu}\gamma_\alpha\frac{i}{(4\pi)^2}\frac{1}{4}\frac{1}{\varepsilon}+\text{finite} \end{aligned} \quad (44)$$

The sum of the γ matrices is, $[\sigma^{\mu\nu},\gamma^\alpha]\otimes\sigma_{\mu\nu}\gamma_\alpha$.

RG running at one loop – Example 2

Using the relations

$$\gamma_\mu \gamma_\nu \gamma_\alpha = g_{\mu\nu}\gamma_\alpha + g_{\nu\alpha}\gamma_\mu - g_{\mu\alpha}\gamma_\nu - i\varepsilon_{\mu\nu\alpha\beta}\gamma_5\gamma^\beta, \quad (45)$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3,\ \varepsilon^{0123} = -\varepsilon_{0123} = +1,$$

we compute as follows

$$\sigma_{\mu\nu}\gamma_\alpha = i\left(+g_{\nu\alpha}\gamma_\mu - g_{\mu\alpha}\gamma_\nu - i\varepsilon_{\mu\nu\alpha\beta}\gamma_5\gamma^\beta\right) \quad (46)$$

$$\gamma_\alpha\sigma_{\mu\nu} = i\left(-g_{\nu\alpha}\gamma_\mu + g_{\mu\alpha}\gamma_\nu - i\varepsilon_{\mu\nu\alpha\beta}\gamma_5\gamma^\beta\right) \quad (47)$$

$$\left[\sigma_{\mu\nu},\gamma_\alpha\right] = i2(g_{\nu\alpha}\gamma_\mu - g_{\mu\alpha}\gamma_\nu) \quad (48)$$

$$\left[\sigma_{\mu\nu},\gamma_\alpha\right]\otimes\sigma^{\mu\nu}\gamma^\alpha = i2(g_{\nu\alpha}\gamma_\mu - g_{\mu\alpha}\gamma_\nu)\otimes i\gamma^\mu\gamma^\nu\gamma^\alpha = -12\gamma^\mu\otimes\gamma_\mu \quad (49)$$

In summary,

$$\text{diagrams with } C_T\mathscr{O}_T \text{ inserted} = iG_T\mu^{2\varepsilon}y_\phi^2(\gamma_\mu\otimes\gamma^\mu)\frac{-6}{(4\pi)^2}\frac{1}{\varepsilon} + \text{finite} \quad (50)$$

RG running at one loop – Example 2

These divergences are cancelled by c.t.:

$$(\gamma_\mu \otimes \gamma^\mu) \qquad 0 = i[Z_{G_V} Z_\psi^2 - 1] G_V \mu^{2\varepsilon} + i G_V \mu^{2\varepsilon} y_\phi^2 \frac{1}{(4\pi)^2} \frac{1}{\varepsilon} + i G_T \mu^{2\varepsilon} y_\phi^2 \frac{-6}{(4\pi)^2} \frac{1}{\varepsilon}$$

$$(\sigma_{\mu\nu} \otimes \sigma^{\mu\nu}) \qquad 0 = i[Z_{G_T} Z_\psi^2 - 1] G_T \mu^{2\varepsilon} + i G_V \mu^{2\varepsilon} y_\phi^2 \frac{-1}{(4\pi)^2} \frac{1}{\varepsilon}$$

$$\Rightarrow \begin{cases} (Z_{G_V} Z_\psi^2 - 1) G_V = \frac{y_\phi^2}{(4\pi)^2} \frac{1}{\varepsilon} (-G_V + 6 G_T) \\ (Z_{G_T} Z_\psi^2 - 1) G_T = \frac{y_\phi^2}{(4\pi)^2} \frac{1}{\varepsilon} G_V \end{cases} \tag{51}$$

Using eq.(12) for β_{G_S} that also applies here and eq.(14), the above gives

$$\beta_{G_V} = \frac{y_\phi^2}{(4\pi)^2} 12 G_T, \; \beta_{G_T} = \frac{y_\phi^2}{(4\pi)^2} 2(G_V + G_T) \tag{52}$$

RG running at one loop – Example 2

In terms of matrix notation, RGEs become

$$\mu\frac{d}{d\mu}\begin{pmatrix} G_V \\ G_T \end{pmatrix} = \frac{2y_\phi^2}{(4\pi)^2}\begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} G_V \\ G_T \end{pmatrix} \tag{53}$$

The matrix on rhs can be diagonalized by a similarity transformation to the eigenvalues and eigenvectors:

$$G_1 = \frac{1}{\sqrt{10}}(G_V + 3G_T),\ G_2 = \frac{1}{\sqrt{5}}(G_V - 2G_T), \tag{54}$$

$$\mu\frac{dG_1}{d\mu} = y_\phi^2 a_1 G_1,\ a_1 = \frac{6}{(4\pi)^2}, \tag{55}$$

$$\mu\frac{dG_2}{d\mu} = y_\phi^2 a_2 G_2,\ a_2 = -\frac{4}{(4\pi)^2}. \tag{56}$$

RG running at one loop – Example 2

Including RG running of y_ϕ in eq.(26), the leading log can be summed as using eq.(28):

$$\frac{G_1(\mu)}{G_1(\mu_0)} = R^{3/5}, \; \frac{G_2(\mu)}{G_2(\mu_0)} = R^{-2/5}, \; R = \frac{y_\phi^2(\mu)}{y_\phi^2(\mu_0)} \tag{57}$$

which translate into the running of the original couplings:

$$G_V(\mu) = \frac{1}{5}\left[\left(2R^{3/5}+3R^{-2/5}\right)G_V(\mu_0)+6\left(R^{3/5}-R^{-2/5}\right)G_T(\mu_0)\right] \tag{58}$$

$$G_T(\mu) = \frac{1}{5}\left[\left(3R^{3/5}+2R^{-2/5}\right)G_T(\mu_0)+\left(R^{3/5}-R^{-2/5}\right)G_V(\mu_0)\right] \tag{59}$$

Conclusion: so long as y_ϕ runs, a nonzero G_V can always develop from a nonzero G_T.

Matching calculation at one loop

We said earlier that matching and RG running can be done independently to various orders in perturbation theory.

Matching: find out new terms in $\mathscr{L}_2$ for EFT$_2$ (IR theory) that account for effects on light fields of a heavy field that appears in $\mathscr{L}_1$ for EFT$_1$ (UV theory) but has been integrated out in $\mathscr{L}_2$ for EFT$_2$.

Continue with our example:
Example 1: heavy scalar

$$\begin{aligned}\mathscr{L}_1(\Phi,\phi,\psi) &= [i\bar{\psi}\partial\!\!\!/\psi - \kappa\bar{\psi}\psi + \cdots] + \left[\frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}M^2\Phi^2 + \cdots\right] \\ &\quad - y_\Phi\bar{\psi}\psi\Phi + \cdots \end{aligned} \tag{60}$$

Light fields: ψ of mass κ, ϕ of mass m.
Heavy field: Φ of mass $M \gg m,\ \kappa$

Matching calculation at one loop – Example 1

We want to increase accuracy in $\psi\psi \to \psi\psi$ scattering amplitude.

This is accomplished by improvement in RGE and matching calc.

For matching at one loop, this requires to improve bilinear and quartic terms in ψ:

$$\mathscr{L}_2(\phi,\psi) \quad = \quad i z_\psi \bar{\psi}\partial\!\!\!/\psi - \kappa\bar{\psi}\psi + \frac{1}{2}G_S\bar{\psi}\psi\bar{\psi}\psi + \cdots \tag{61}$$

Our notations are a bit messy: we sometimes write explicitly renormalization constants or c.t. but sometimes not.

Matching calculation at one loop – Example 1

How to do matching at one loop?

- Choose a one-particle-irreducible function of light fields that helps to determine terms in $\mathscr{L}_2$.
- Compute the function at one loop in EFT_1 involving a heavy field Φ of mass M, and fix all relevant c.t. using mass-independent scheme. We get answer A_1.
- Compute the function at one loop in EFT_2 involving an effective interaction that arises from integrating out Φ, and fix all relevant c.t. using the same scheme. We get answer A_2.
- Take the difference $A_1 - A_2$ and set the scale $\mu = M$, and put the answer back into $\mathscr{L}_2$.

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Matching calculation at one loop – Example 1

- For example, to compute the scattering amplitude $\psi\psi \to \psi\psi$ at one loop in EFT$_2$, we have to do matching in both z_ϕ, which affects normalization, and G_S, which contributes to the part of the amplitude due to integrated out Φ.
- Matching calculation of z_ϕ.
  In EFT$_1$ the self-energy diagram of ψ due to Yukawa coupling with Φ gives

$$(-iy_\Phi)^2\mu^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i}{\not{k}+\not{p}-\kappa}\frac{i}{k^2-M^2}$$

$$= \quad y_\Phi^2\int dx\,\mu^{2\varepsilon}\int\frac{d^d\ell}{(2\pi)^d}\frac{x\not{p}+\kappa}{[\ell^2-\Delta+i0^+]^2},\ \Delta=(1-x)\kappa^2+xM^2$$

(Feynman diagram on page 10)

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\end{aligned}
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  (Feynman diagram on page 10)

Matching calculation at one loop – Example 1

- We want the part $\propto \not{p}$ whose coefficient is

$$\frac{iy_\Phi^2}{(4\pi)^2}\int dx\, x\left[\frac{1}{\bar{\varepsilon}}+\ln\frac{\mu^2}{\Delta - i0^+}\right],\quad \frac{1}{\bar{\varepsilon}}=\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi)$$

- In modified minimal subtraction ($\overline{\text{MS}}$), we cancel the $1/\bar{\varepsilon}$ term by c.t. $(Z_\psi - 1)i\not{p}$, leaving with us the finite piece:

$$\frac{iy_\Phi^2}{(4\pi)^2}\left[\frac{1}{2}\ln\frac{\mu^2}{M^2}-\int dx\, x\ln\left(x+(1-x)\delta\right)\right],\quad \delta=\frac{\kappa^2}{M^2}$$

which can be expanded systematically in the small parameter δ:

$$\frac{iy_\Phi^2}{(4\pi)^2}\left[\frac{1}{2}\ln\frac{\mu^2}{M^2}+\frac{1}{4}-\frac{1}{4}\delta^2(2\ln\delta+1)+\cdots\right]$$

Exercise – derive the above expansion.

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Matching calculation at one loop – Example 1

- In EFT$_2$ the one loop formed by G_S coupling does not contribute a $\not p$ term to the self-energy of ψ.
- This difference between EFT$_1$ and EFT$_2$ is then amended by attaching to $\mathscr{L}_2$ a term:

$$(z_\psi - 1) i \bar{\psi} \not\partial \psi, \tag{62}$$

where at $\mu = M$

$$z_\psi(M) - 1 = \frac{y_\Phi^2}{(4\pi)^2} \frac{1}{4} [1 + \cdots]. \tag{63}$$

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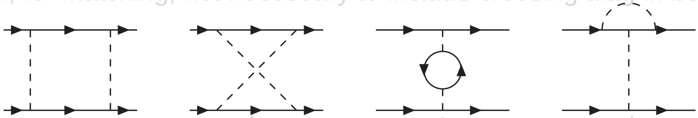
Matching calculation at one loop – Example 1

Matching calculation of G_S.

- In EFT_1 compute 1-loop contri. to $\psi\psi \to \psi\psi$ due to Yukawa coupling of Φ.
- Again, for matching, not necessary to include crossing diag in both EFTs.



Focus on momentum-indept terms. But we keep a mass κ for ψ to avoid IR divergence. The first two diagrams are finite:

$$\text{diagram a} = (-iy_\Phi \mu^\varepsilon)^4 \int \frac{d^dk}{(2\pi)^d} \bar{u}_3 \frac{i}{\not{k} - \kappa} u_1 \, \bar{u}_4 \frac{i}{-\not{k} - \kappa} u_2 \left[\frac{i}{k^2 - M^2}\right]^2 \quad (64)$$

$$\text{diagram b} = (-iy_\Phi \mu^\varepsilon)^4 \int \frac{d^dk}{(2\pi)^d} \bar{u}_3 \frac{i}{\not{k} - \kappa} u_1 \, \bar{u}_4 \frac{i}{+\not{k} - \kappa} u_2 \left[\frac{i}{k^2 - M^2}\right]^2 \quad (65)$$

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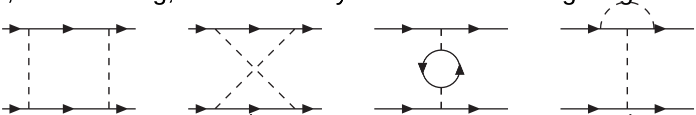

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Matching calculation at one loop – Example 1

- Their sum is simpler:

$$\text{diagrams (ab)} = y_\Phi^4 \mu^{2\varepsilon} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}_3(\not{k}+\kappa)u_1\bar{u}_4(2\kappa)u_2}{(k^2-\kappa^2)^2[k^2-M^2]^2}$$
$$= y_\Phi^4 \mu^{2\varepsilon} 2\kappa^2 \bar{u}_3 u_1 \bar{u}_4 u_2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\kappa^2)^2[k^2-M^2]^2} \quad (66)$$

where we reserve $\mu^{2\varepsilon}$ associated with G_S.

- Compute the loop integral:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\kappa^2)^2[k^2-M^2]^2}$$
$$= \int_0^1 dx\, 6x(1-x) \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-\Delta+i0^+]^4} \qquad \delta = \frac{\kappa^2}{M^2}$$
$$= \frac{i}{(4\pi)^2} \int_0^1 dx \frac{x(1-x)}{\Delta^2} = \frac{i}{(4\pi)^2} \frac{1}{M^4} \int_0^1 dx \frac{x(1-x)}{[x+(1-x)\delta]^2}, \quad (67)$$

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Matching calculation at one loop – Example 1

- We want to expand in small δ.
  It is not possible to expand the integrand directly.
  You may appeal to *Mathematica* etc. But sometimes you have to do it yourself when softwares cannot do it well.
- There is a systematical way to do so here by first finishing integration with fractioning,

$$\frac{x(1-x)}{(x+a)^2} = \frac{-(x+a)^2+(1+2a)(x+a)-a(1+a)}{(x+a)^2}, \; a=\frac{\delta}{1-\delta}$$

$$\int_0^1 dx \frac{x(1-x)}{[x+(1-x)\delta]^2} = \frac{1}{(1-\delta)^2}\left[-1+(1+2a)\ln\frac{1+a}{a} - a(1+a)\left(\frac{1}{a}-\frac{1}{1+a}\right)\right]$$

and then expanding in δ.

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Matching calculation at one loop – Example 1

- In summary, the first terms are

$$\text{diag (ab)} = \frac{iy_\Phi^4}{(4\pi)^2}\frac{2\kappa^2}{M^4}\mu^{2\varepsilon}\bar{u}_3 u_1 \bar{u}_4 u_2 \left[(-2-\ln\delta)+(-4-4\ln\delta)\delta+\cdots\right] \qquad (68)$$

- To compute diagram c, compute first self-energy of Φ due to ψ loop:

$$\begin{aligned} & i\Sigma_\Phi(p^2) \\ = & -(-iy_\Phi\mu^\varepsilon)^2\int\frac{d^dk}{(2\pi)^d}\mathrm{tr}\frac{i}{\not{k}-\kappa}\frac{i}{\not{k}+\not{p}-\kappa} \\ = & -\frac{iy_\Phi^2}{(4\pi)^2}12\int_0^1[\kappa^2-x(1-x)p^2]\left[\frac{1}{\bar\varepsilon}+\frac{1}{3}+\ln\frac{\mu^2}{\kappa^2-x(1-x)p^2}\right] \\ = & -\frac{iy_\Phi^2}{(4\pi)^2}\left\{2[6\kappa^2-p^2]\frac{1}{\bar\varepsilon}+12\int_0^1[\kappa^2-x(1-x)p^2]\left[\frac{1}{3}+\ln\frac{\mu^2}{\kappa^2-x(1-x)p^2}\right]\right\} \end{aligned} \qquad (69)$$

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Matching calculation at one loop – Example 1

- The $1/\bar{\varepsilon}$ term is cancelled by c.t., so that as $p^2 \to 0$ we have

$$i\left[\Sigma_\Phi(p^2)+\text{c.t.}\right]_{p^2=0} = -\frac{iy_\Phi^2}{(4\pi)^2}4\kappa^2\left[1+3\ln\frac{\mu^2}{\kappa^2}\right] \tag{70}$$

Caution: likely illegitimate to drop p^2 with respect to κ^2.

$$\begin{aligned}\text{diag c and c.t.} &= (-iy_\Phi\mu^\varepsilon)^2\bar{u}_3u_1\bar{u}_4u_2\left(\frac{i}{-M^2}\right)^2 i\left[\Sigma_\phi(p^2)+\text{c.t.}\right]_{p^2=0} \\ &= -\frac{iy_\Phi^4}{(4\pi)^2}\frac{4\kappa^2}{M^4}\mu^{2\varepsilon}\bar{u}_3u_1\bar{u}_4u_2\left[1+3\ln\frac{\mu^2}{\kappa^2}\right]\end{aligned} \tag{71}$$

Matching calculation at one loop – Example 1

- To compute diag d, compute first 1-loop $\Phi\bar{\psi}\psi$ vertex due to Φ Yukawa coupling.
- UV div is independent of external momenta, but again the dropped p^2 term is of the same order as the kept κ^2 term.

$$
\begin{aligned}
-iy_\Phi \mu^\varepsilon V_{\Phi\bar{\psi}\psi}(0,0) &= (-iy_\Phi\mu^\varepsilon)^3 \int \frac{d^d k}{(2\pi)^d} \frac{i}{\not{k}-\kappa} \frac{i}{\not{k}-\kappa} \frac{i}{k^2-M^2} \\
&= y_\Phi^3 \mu^{3\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{k^2+\kappa^2}{[k^2-\kappa^2]^2[k^2-M^2]} \\
&= y_\Phi^3 \mu^\varepsilon \frac{i}{(4\pi)^2} \int dx \left[\frac{1}{\bar{\varepsilon}} + \ln\frac{\mu^2}{\Delta} - \frac{2(1-x)\kappa^2}{\Delta} \right] \qquad (72)
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Matching calculation at one loop – Example 1

- The $1/\bar{\varepsilon}$ term is removed by c.t. for the vertex, so that

$$
\begin{aligned}
& -iy_\Phi \mu^\varepsilon V_{\Phi\bar{\psi}\psi}(0,0) + \text{c.t.} \qquad \delta = \frac{\kappa^2}{M^2} \\
= \quad & iy_\Phi \mu^\varepsilon \frac{y_\Phi^2}{(4\pi)^2} \int dx \left[\ln \frac{\mu^2}{M^2} - \ln[x + (1-x)\delta] - \delta \frac{2(1-x)}{x + (1-x)\delta} \right] \\
= \quad & iy_\Phi \mu^\varepsilon \frac{y_\Phi^2}{(4\pi)^2} \left[\ln \frac{\mu^2}{M^2} + 1 + (2 + 3\ln\delta)\delta + \cdots \right] \qquad (73)
\end{aligned}
$$

Including a factor of 2, we have

$$
\begin{aligned}
& \text{diag d and c.t.} \\
= \quad & 2 \times (-iy_\Phi \mu^\varepsilon) \frac{i}{-M^2} \bar{u}_3 u_1 \bar{u}_4 u_2 \left[-iy_\Phi \mu^\varepsilon V_{\Phi\bar{\psi}\psi}(0,0) + \text{c.t.} \right] \\
= \quad & -\frac{iy_\Phi^4}{(4\pi)^2} \mu^{2\varepsilon} \bar{u}_3 u_1 \bar{u}_4 u_2 \frac{2}{M^2} \left[\ln \frac{\mu^2}{M^2} + 1 + \frac{\kappa^2}{M^2} \left(2 + 3\ln \frac{\kappa^2}{M^2} \right) + \cdots \right] \qquad (74)
\end{aligned}
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Matching calculation at one loop – Example 1

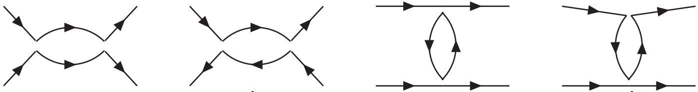
In summary, the leading one-loop renormalized contribution to $\psi\psi \to \psi\psi$ in EFT_1 is

$$\begin{aligned}&\text{renor. one-loop EFT}_1 \text{ diag. for } \psi\psi \to \psi\psi \text{ due to } \Phi \text{ exchange}\\ =\ &\frac{i2y_\Phi^4}{(4\pi)^2 M^2}\mu^{2\varepsilon}\bar{u}_3 u_1 \bar{u}_4 u_2 \left\{-1-\ln\frac{\mu^2}{M^2}+\delta\left(-6-6\ln\frac{\mu^2}{M^2}+2\ln\delta\right)+\cdots\right\} \quad (75)\end{aligned}$$

where it is actually unclear if the $O(\delta)$ terms are complete.

Matching calculation at one loop – Example 1

- In EFT_2 compute the one-loop contribution to $\psi\psi \to \psi\psi$ due to effective G_S coupling.



- All diagrams are now UV divergent.

$$\text{diagram a} = (iG_S\mu^{2\varepsilon})^2 \int \frac{d^dk}{(2\pi)^d} \bar{u}_3 \frac{i}{\not{k} - \kappa} u_1 \bar{u}_4 \frac{i}{-\not{k} - \kappa} u_2 \quad (76)$$

$$\text{diagram b} = (iG_S\mu^{2\varepsilon})^2 \int \frac{d^dk}{(2\pi)^d} \bar{u}_3 \frac{i}{\not{k} - \kappa} u_1 \bar{u}_4 \frac{i}{+\not{k} - \kappa} u_2 \quad (77)$$

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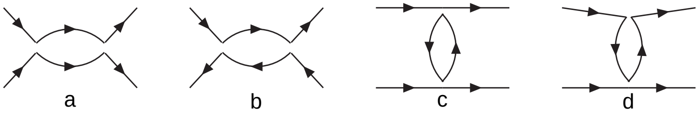

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Matching calculation at one loop – Example 1

- Their sum is

$$\begin{aligned} \text{diagrams (ab)} &= G_S^2 \mu^{2\varepsilon} \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}_3 (\not k + \kappa) u_1 \bar{u}_4 2\kappa u_2}{[k^2 - \kappa^2]^2} \\ &= G_S^2 \mu^{2\varepsilon} \frac{i}{(4\pi)^2} 2\kappa^2 \bar{u}_3 u_1 \bar{u}_4 u_2 \left[\frac{1}{\bar{\varepsilon}} + \ln \frac{\mu^2}{\kappa^2} \right] \end{aligned} \quad (78)$$

- Diagram c is identical in two EFTs in our approximation.

$$\begin{aligned} \text{diagram d} &= 2 \times (i G_S \mu^{2\varepsilon})^2 \int \frac{d^d k}{(2\pi)^d} \bar{u}_3 \frac{i}{\not k - \kappa} \frac{i}{\not k - \kappa} u_1 \bar{u}_4 u_2 \\ &= G_S^2 \mu^{2\varepsilon} \bar{u}_3 u_1 \bar{u}_4 u_2 \frac{i}{(4\pi)^2} 2\kappa^2 \left[\frac{3}{\bar{\varepsilon}} + 3 \ln \frac{\mu^2}{\kappa^2} + 1 \right] \end{aligned} \quad (79)$$

- Absorbing the $1/\bar{\varepsilon}$ terms by c.t. for G_S coupling in $\mathscr{L}_2$, we are left with

renormalized 1-loop EFT$_2$ diagrams for $\psi\psi \to \psi\psi$ due to G_S coupling

$$= G_S^2 \mu^{2\varepsilon} \frac{i}{(4\pi)^2} 2\kappa^2 \bar{u}_3 u_1 \bar{u}_4 u_2 \left\{ -1 - 2 \ln \frac{\mu^2}{\kappa^2} \right\} \quad (80)$$

Matching calculation at one loop – Example 1

- Their sum is

$$\text{diagrams (ab)} = G_S^2 \mu^{2\varepsilon} \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\bar{u}_3(\not{k} + \kappa) u_1 \bar{u}_4 2\kappa u_2}{[k^2 - \kappa^2]^2}$$

$$= G_S^2 \mu^{2\varepsilon} \frac{i}{(4\pi)^2} 2\kappa^2 \bar{u}_3 u_1 \bar{u}_4 u_2 \left[\frac{1}{\bar{\varepsilon}} + \ln \frac{\mu^2}{\kappa^2}\right] \quad (78)$$

- Diagram c is identical in two EFTs in our approximation.

$$\text{diagram d} = 2 \times (iG_S \mu^{2\varepsilon})^2 \int \frac{d^d k}{(2\pi)^d} \bar{u}_3 \frac{i}{\not{k} - \kappa} \frac{i}{\not{k} - \kappa} u_1 \bar{u}_4 u_2$$

$$= G_S^2 \mu^{2\varepsilon} \bar{u}_3 u_1 \bar{u}_4 u_2 \frac{i}{(4\pi)^2} 2\kappa^2 \left[\frac{3}{\bar{\varepsilon}} + 3\ln \frac{\mu^2}{\kappa^2} + 1\right] \quad (79)$$

- Absorbing the $1/\bar{\varepsilon}$ terms by c.t. for G_S coupling in $\mathscr{L}_2$, we are left with

renormalized 1-loop EFT$_2$ diagrams for $\psi\psi \to \psi\psi$ due to G_S coupling

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Matching calculation at one loop – Example 1

- The difference

$$\mathrm{EFT}_1 - \mathrm{EFT}_2$$

gives the correction to G_S upon using $G_S = y_\Phi^2/M^2$:

$$G_S(\mu) = \frac{y_\Phi^2}{M^2} - \frac{2y_\Phi^4}{(4\pi)^2 M^2}\left\{1 + \ln\frac{\mu^2}{M^2} + \delta\left(5 + 4\ln\frac{\mu^2}{M^2}\right)\right\} \quad (81)$$

which should be included in $\mathscr{L}_2$ as a consequence of matching.

- Comments –
  - No $\ln\kappa^2$ singularity appears in the matching result as expected: IR physics is not changed.
  - To avoid large log, we should set $\mu = M$ in matching.
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Summary on EFT calculations

- EFT is as good as a renormalizable theory so long as we are content with finite accuracy required by experiments.
- Our working QFT is a tower of EFTs.
- In top-down approach:
  - From high to low scales, a heavy field is integrated out at the border of two consecutive EFTs.
    Do matching calculation so that its effects on light fields are correctly reproduced.
    Set $\mu = M$, mass of the heavy field, to avoid large log.
  - Within one EFT, do RG running from $\mu = M$ to m, typical scale of a process under consideration.
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